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Nearly Kirkman triple systems of order 18 and Hanani triple systems of order 19

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ABSTRACT

A Hanani triple system of order $6n+1$, HATS$(6n+1)$, is a decomposition of the complete graph K_{6n+1} into $3n$ sets of $2n$ disjoint triangles and one set of n disjoint triangles. A nearly Kirkman triple system of order $6n$, NKTS$(6n)$, is a decomposition of $K_{6n} - F$ into $3n-1$ sets of $2n$ disjoint triangles; here F is a one-factor of K_{6n}. The Hanani triple systems of order $6n+1$ and the nearly Kirkman triple systems of order $6n$ can be classified using the classification of the Steiner triple systems of order $6n+1$. This is carried out here for $n=3$: There are 3787983639 isomorphism classes of HATS(19)s and 25328 isomorphism classes of NKTS(18)s. Several properties of the classified systems are tabulated. In particular, seven of the NKTS(18)s have orthogonal resolutions, and five of the HATS(19)s admit a pair of resolutions in which the almost parallel classes are orthogonal.

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1. Introduction

A *Steiner triple system* (STS) is a pair $(X, \mathcal{B})$, where X is a finite set of *points* and $\mathcal{B}$ is a collection of 3-subsets of points, called *blocks* or *triples*, such that every 2-subset of points occurs in exactly one block. The size of the point set is the *order* of the STS; an STS of order v is commonly denoted by STS(v). STSs exist exactly for orders $v \equiv 1, 3 \pmod 6$. For surveys of STSs and their properties, see [3,5].

A *partial parallel class* (PPC) is a set of disjoint blocks. A PPC with $\frac{v}{3}$ blocks is a *parallel class*, and a PPC with $\frac{v-1}{3}$ blocks is an *almost parallel class* (APC). A *resolution* of an STS is a partition of the blocks into parallel classes. An STS that has at least one resolution is *resolvable*, and an STS together with a resolution is a *Kirkman triple system* (KTS). KTSs exist exactly for orders $v \equiv 3 \pmod 6$.

Two KTSs are *isomorphic* if there is a bijection between their point sets that maps parallel classes (and the blocks therein) onto parallel classes. Such a mapping from a KTS onto itself is an *automorphism*; all automorphisms form the *automorphism group* of the KTS. KTSs have been classified up to order 15; up to isomorphism, there exist one KTS(9) and seven KTS(15)s with four underlying STS(15)s [3].

There are several types of 'resolvable' structures closely related to KTSs, two of which are considered in this work. Suppose that $v \equiv 1 \pmod 6$ and consider an STS(v). The maximum possible number of disjoint APCs is $(v-1)/2$. An STS(v) with

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the blocks partitioned into $(v-1)/2$ APCs and with the remaining $(v-1)/6$ blocks forming a PPC is a *Hanani triple system* of order v, HATS(v). HATSs exist exactly for orders $v \equiv 1 \pmod 6$, $v \geq 19$ [20]. In graph-theoretic terms, a HATS(v) is a decomposition of the complete graph K_v into $(v-1)/2$ sets of $(v-1)/3$ disjoint triangles and one set of $(v-1)/6$ disjoint triangles. The existence of a HATS(v) implies that the underlying STS(v) has chromatic index $(v+1)/2$, cf. [4,20].

Now suppose that $v \equiv 0 \pmod 6$. A *nearly Kirkman triple system* of order v, NKTS(v), is a partial Steiner triple system that covers all but $v/2$ disjoint 2-subsets, whose blocks partition into parallel classes. NKTSs exist exactly for orders $v \equiv 0 \pmod 6$, $v \geq 18$, see [1,2,12,16]. In graph-theoretic terms, an NKTS(v) is a decomposition of $K_v - F$ into 2-factors consisting of triangles, where F is a one-factor of K_{6n}.

A *group divisible design* with block size k, or k-GDD, of type g^a is a triple $(X, \mathcal{G}, \mathcal{B})$, where X is a set of ga points, $\mathcal{G}$ is a partition of X into a subsets of size g, called *groups*, and $\mathcal{B}$ is a collection of k-subsets (blocks) of points, such that every 2-subset of points occurs in exactly one block or one group, but not both. An STS(v) is therefore a 3-GDD of type 1^v. A 3-GDD of type 2^{3v} whose blocks partition into parallel classes is precisely an NKTS$(6v)$.

The computational problem of classifying HATS$(6n+1)$s and NKTS$(6n)$s from a classification of STS$(6n+1)$s is considered in Section 2. The classification results for HATS(19)s and NKTS(18)s are presented in Section 3. There are 3787983639 and 25328 isomorphism classes of such systems, respectively. A partial classification of NKTS(18)s has been carried out previously in [13]. Several properties of the classified systems are tabulated and discussed. Of particular interest, seven of the NKTS(18)s have orthogonal resolutions.

2. Classification

There are three major approaches for classifying resolutions of designs: parallel class by parallel class, point by point, and via the underlying designs [8, Section 6.3]. The last of these is adopted here, because HATS(19)s and NKTS(18)s are directly related to STS(19)s.

2.1. Hanani triple systems

The design underlying a HATS(v) is an STS(v). To classify the HATS(v)s, carry out the following computation for each STS(v). Determine all APCs by formulating the associated instance of the exact cover problem, as described in [4], and use the `libexact` software [11] to solve it. Then form a graph G with one vertex for each APC and an edge between two vertices exactly when the corresponding APCs do not have a block in common. Then search for all cliques of size $(v-1)/2$ using, for example, the `Cliquer` software [15]. For each such clique (solution), check whether the remaining $(v-1)/6$ blocks are disjoint. The final check can be eliminated, as follows.

Lemma 1. *There is a clique in G of size $(v-1)/2$ in which no two APCs have the same missing point if and only if there is a* HATS(v).

Proof. Consider a HATS(v), $(X, \mathcal{B})$; let M be the set of points missed by at least one APC. Because each of the $(v-1)/2$ APCs misses exactly one point, the number of points *not* missed can be bounded as $|X \setminus M| \geq (v+1)/2$. For each point $x \in X \setminus M$, all blocks containing x appear in the APCs. Therefore the points appearing in blocks of the PPC of size $(v-1)/6$ are exactly the points of M. So $|M| = (v-1)/2$, and different APCs miss different points. The converse is straightforward. □

Using Lemma 1 one can modify the graph instance to remove edges that correspond to APCs with the same missing point. Then there is no need to check the solutions corresponding to cliques further.

2.2. Nearly Kirkman triple systems

An NKTS(v) can be used to construct an STS$(v+1)$ in the following way. The blocks underlying the NKTS(v) form a 3-GDD of type $2^{v/2}$. From such a group divisible design, an STS$(v+1)$ is obtained by including one more point p in the point set, and adding the block $\{p, p', p''\}$ for each pair of points $\{p', p''\}$ that forms a group of the GDD. Reversing this construction gives the desired classification approach.

To classify the NKTS(v)s, carry out the following computation for each STS$(v+1)$ $(X, \mathcal{B})$ and for each point $p \in X$. Delete the point p and each block containing p. Among the remaining blocks, determine all parallel classes, using instances of the exact cover problem. Then determine all partitions of the blocks into parallel classes, also using exact cover [11].

2.3. Isomorph rejection

Isomorphs must be removed from the collection of systems so obtained. The definitions of isomorphic HATSs and NKTSs are analogous to the definition of isomorphic KTSs.

Isomorph rejection is simplified by the fact that two HATSs or NKTSs can be isomorphic only if the underlying designs are isomorphic. Moreover, the automorphism group must be a subgroup of the automorphism group of the underlying design. The graph isomorphism software *nauty* [14] is a practical tool for these computations.

Table 1
Automorphism group orders for HATS(19)s.

$\lvert\mathrm{Aut}(\mathcal{S})\rvert$	$\lvert\mathrm{Aut}(\mathcal{T})\rvert$	#
1	1	3787950179
2	1	26387
3	1	1998
3	3	4718
4	1	324
6	1	16
6	3	4
8	1	1
9	3	3
9	9	6
12	1	1
18	1	1
18	3	1

Table 2
Automorphism group orders for NKTS(18)s.

$\lvert\mathrm{Aut}(\mathcal{S})\rvert$	$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	$\lvert\mathrm{Aut}(\mathcal{T})\rvert$	#
1	1	1	24882
2	2	1	4
2	2	2	216
3	1	1	2
3	3	3	103
4	4	1	2
4	4	2	27
4	4	4	53
6	6	3	3
6	6	6	12
8	8	2	4
12	12	6	2
16	16	4	2
18	18	2	1
18	18	6	2
18	18	18	2
32	32	16	3
144	144	4	2
144	144	6	2
144	144	36	4

3. Results

A classification of the HATS(19)s and the NKTS(18)s with the approaches discussed here was carried out twice, with different programs in different computing environments. The catalogue of STS(19)s produced in [7,9] was used as a starting point for the computations. An equivalent of at least 2 years of CPU time on contemporary personal computers was needed for each of the classification programs.

3.1. Numbers and automorphism groups

Theorem 1. *The number of isomorphism classes of* HATS(19)*s is* 3787983639, *obtained from* 2894565584 *underlying* STS(19)*s. The number of isomorphism classes of* NKTS(18)*s is* 25328, *obtained from* 25164 STS(19)*s.*

In Table 1, the number of HATS(19)s is aggregated by the order of the automorphism group of the underlying STS(19), $\mathcal{S}$, and of the HATS, $\mathcal{T}$; and in Table 2, the number of NKTS(18)s is aggregated by the order of the automorphism group of the related STS(19), $\mathcal{S}$, of the underlying 3-GDD, $\mathcal{D}$, and of the NKTS, $\mathcal{T}$. There is only one entry in Table 2 for which the order of the STS(19) differs from the order of the 3-GDD.

The distribution of the number of isomorphism classes of HATSs and NKTSs obtainable from a single STS is displayed in Table 3; the number is denoted by N. It turns out that all NKTSs arising from a given STS have the same underlying GDD.

Further properties of the classified systems are studied. Due to the small number of NKTS(18)s, these systems can easily be saved and processed separately.

Table 3
Number of nonisomorphic systems.

N	# (HATS)	# (NKTS)
1	2213217547	25029
2	521527151	123
3	121045078	3
4	28952386	5
5	7211930	2
6	1882965	1
7	515585	
8	147887	1
9	44119	
10	13926	
11	4531	
12	1614	
13	514	
14	205	
15	90	
16	33	
17	15	
18	3	
19	3	
22	1	
23	1	

3.2. Subsystems

Lemma 2. *A* HATS(19) *cannot embed a* KTS(9).

Proof. The point set P of an STS(9) embedded in an STS(19) intersects each block of the STS(19) in either 1 or 3 points. Hence a parallel class of the KTS(9) cannot be extended to an APC of the HATS(19) because there are no blocks that intersect P in 0 points. □

Because no HATS(7) exists, trivially a HATS(19) cannot embed a HATS(7). For NKTS(18)s, subdesigns of order 7 and 9 cannot arise, whether resolved or not, as is shown next.

Lemma 3. *An* NKTS(18) *cannot embed an* STS(7).

Proof. Suppose to the contrary that an NKTS(18) embeds an STS(7). An NKTS(18) has 8 parallel classes of 6 blocks each. Because the blocks of an STS(7) have pairwise nonempty intersection, each of the 7 blocks occurs in a unique parallel class. Then one parallel class does not contain any block from the STS(7), and hence each of its blocks contains exactly one point from the STS(7). However, this parallel class has 6 blocks, so it does not include all points of the STS(7), a contradiction. □

Lemma 4. *A 3-GDD of type* 2^v *that contains an* STS(v) *cannot have a parallel class. Consequently, an* NKTS$(2v)$ *cannot embed an* STS(v).

Proof. An STS(v) exists, so v is odd. Consider a 3-GDD of type 2^v that embeds an STS(v). Let P be the set of v points of the STS(v), and let Q be the set of v points of the 3-GDD not in the STS(v). Let ρ_{ij} be the number of pairs having i points from P and j from Q that appear in blocks of the 3-GDD, and let τ_{ij} be the number of blocks of the 3-GDD having i points from P and j from Q. Now $\rho_{20} = \binom{v}{2}$, and because P induces an STS(v), $\tau_{30} = \binom{v}{2}/3$. Because $3\tau_{30} + \tau_{21} = \rho_{20}$, we find that $\tau_{21} = 0$. Then because $\rho_{11} = v(v-1)$ and $\rho_{11} = 2(\tau_{21} + \tau_{12})$, it follows that $\tau_{12} = \binom{v}{2}$. Finally, because $\rho_{02} = \binom{v}{2}$ and $3\tau_{03} + \tau_{12} = \rho_{02}$, we find that $\tau_{03} = 0$. Hence every block involves an even number of points in Q. Because v is odd, no subset of the blocks can contain each element of Q exactly once, and hence there is no parallel class of blocks. □

The situation is different for HATS(19)s. A classification [10] of STS(19)s with subsystems of order 7 or 9 can be used to determine whether such designs underlie any HATS(19).

Lemma 5. *Among the* HATS(19)*s,*

1. *exactly* 450 *embed both an* STS(7) *and an* STS(9)*;*
2. *exactly* 23864 *embed an* STS(9) *but do not embed an* STS(7)*; and*
3. *exactly* 15917314 *embed an* STS(7) *but do not embed an* STS(9).

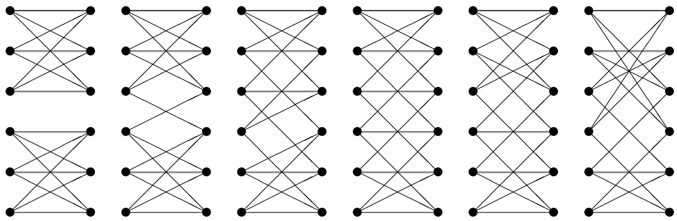

Fig. 1. The six bipartite cubic graphs on 12 points.

Table 4
Combinations of types of pairs.

Type	#	Type	#	Type	#	Type	#
12345	6	1345	5	235	14	3456	8432
123456	61	13456	30	2356	235	346	3
12346	5	1346	5	245	2	35	5
12356	2	146	1	2456	3	356	163
1245	1	2345	1951	246	9	456	7
1246	2	23456	13627	256	2		
134	1	2346	2	345	754		

3.3. Types of parallel class pairs

Two parallel classes of an NKTS(18) form a bipartite cubic graph with one vertex for each block and edges between intersecting blocks. There are, up to isomorphism, six possible such graphs, listed in [13, Table 1] and shown here in Fig. 1 in the same order.

The combinations of types of pairs of parallel classes in the classified NKTS(18)s are presented in Table 4. The results corroborate the main result of [13] that there are 119 NKTS(18)s with at least one parallel class of Type 1 (a total of 135 NKTS(18)s were found in [13]).

A system that contains all possible types of pairs of parallel classes is *type-heterogeneous* [6]. There are 61 isomorphism classes of type-heterogeneous NKTS(18)s. A system with only one type of pairs of parallel classes is *type-uniform* [6]. There is no type-uniform NKTS(18). The smallest number of types of pairs of parallel classes is 2 (the smallest previously known number [13] was 3); there are five such NKTS(18)s, all of which have parallel class pairs of Types 3 and 5.

3.4. Orthogonal resolutions

Given a design and two resolutions into (partial) parallel classes, the resolutions are *orthogonal* if any pair of (partial) parallel classes, one from each resolution, intersect in at most one block. A design that has two orthogonal resolutions is *doubly resolvable*. A HATS(19) is *almost doubly resolvable* if there are two resolutions into nine APCs and one partial parallel class so that the almost parallel classes of one are orthogonal to those of the other; but the requirement that the single short PPC be also orthogonal is dropped. When the short PPCs in the two HATS(19)s are, in fact, the same, an almost double resolution consists of two orthogonal resolutions of $K_{19} - G$ into APCs, where G consists of three vertex-disjoint triangles.

For each HATS(19) and NKTS(18), the number N of (almost) parallel classes orthogonal to the (almost) parallel classes of the given resolution is collected in Table 5 (cf. Column P_6 in the table on [13, p. 91]). To find doubly resolvable NKTS(18)s or almost doubly resolvable HATS(19)s, it suffices to check the cases $N \geq 8$ for NKTS(18)s and the cases $N \geq 9$ for HATS(19)s.

The smallest doubly resolvable NKTS(v) that was previously known is an NKTS(24) found by Smith [18]; indeed relatively few orders have been settled affirmatively [17,19].

Theorem 2. *There are seven doubly resolvable* NKTS(18)*s, up to isomorphism.*

Proof. The systems are listed as 8×8 squares below. Columns partition the blocks into one resolution, while rows specify the orthogonal resolution. The 7 classes of NKTS(18)s with an orthogonal resolution partition into 3 self-orthogonal NKTS(18)s – that is, the orthogonal resolution is isomorphic to the original resolution – and 2 pairs of nonisomorphic mutually orthogonal resolutions. The orders of the automorphism groups for each NKTS(18), $\mathcal{T}$, the underlying GDD, $\mathcal{D}$, and the enclosing STS, $\mathcal{S}$, are all given.

#N1: $|\mathrm{Aut}(\mathcal{T})| = 16$
$(0, 4, 0, 0, 20, 4)$

$|\mathrm{Aut}(\mathcal{S})| = 32$, $|\mathrm{Aut}(\mathcal{D})| = 32$
#N1: $|\mathrm{Aut}(\mathcal{T})| = 16$, $(0, 4, 0, 0, 20, 4)$

		agh	bkm	ejp	fir	clq	dno
		bln	aij	cho	dgq	ekr	fmp
acd	boq			gmr	hkp	fjn	eil
bpr	aef			inq	jlo	dhm	cgk
fgl	dik	emq	cnr			aop	bhj
ehn	cjm	fko	dlp			bgi	aqr
imo	gnp	djr	fhq	akl	bce		
jkq	hlr	cip	ego	bdf	amn		

#N2: $|\mathrm{Aut}(\mathcal{T})| = 16$
$(0, 4, 0, 16, 8, 0)$

#N2: $|\mathrm{Aut}(\mathcal{S})| = 32$, $|\mathrm{Aut}(\mathcal{D})| = 32$
$|\mathrm{Aut}(\mathcal{T})| = 16$, $(0, 4, 0, 16, 8, 0)$

		dlr	cnq	fgm	eik	aop	bhj
		fkp	emo	dhn	cjl	bgi	aqr
egq	cho			akl	bpr	fjn	dim
fir	djp			boq	amn	ehl	cgk
jko	ilq	agh	bdf			cmr	enp
hmp	gnr	bce	aij			dkq	flo
acd	bkm	ino	glp	ejr	fhq		
bln	aef	jmq	hkr	cip	dgo		

#N3: $|\mathrm{Aut}(\mathcal{T})| = 16$
$(0, 20, 0, 0, 4, 4)$

#N3: $|\mathrm{Aut}(\mathcal{S})| = 32$, $|\mathrm{Aut}(\mathcal{D})| = 32$
$|\mathrm{Aut}(\mathcal{T})| = 16$, $(0, 20, 0, 0, 4, 4)$

		cnr	dlq	fgm	eik	aop	bhj
		emp	fko	dhn	cjl	bgi	aqr
egq	cho			akl	bpr	fjn	dim
fir	djp			boq	amn	ehl	cgk
jmo	inq	agh	bce			dkr	flp
hkp	glr	bdf	aij			cmq	eno
acd	bkm	ilo	gnp	ejr	fhq		
bln	aef	jkq	hmr	cip	dgo		

#N4: $|\mathrm{Aut}(\mathcal{T})| = 6$
$(0, 6, 6, 4, 12, 0)$

#N5: $|\mathrm{Aut}(\mathcal{S})| = 6$, $|\mathrm{Aut}(\mathcal{D})| = 6$
$|\mathrm{Aut}(\mathcal{T})| = 6$, $(0, 12, 0, 12, 0, 4)$

abc	jor		elp	dim		ghq	fkn
gjl	ade		cnr	foq		bik	hmp
hno	ckm	afg		bpr	dlq		eij
	bnq	eko	ahi		fjp	lmr	cdg
fir		cpq	gmo	akl	beh	djn	
	gip	dhr	jkq		amn	cef	blo
emq		iln	bdf	chj	gkr	aop	
dkp	fhl	bjm		egn	cio		aqr

#N6: $|\mathrm{Aut}(\mathcal{T})| = 3$
$(0, 6, 9, 3, 6, 4)$

#N7: $|\mathrm{Aut}(\mathcal{S})| = 3$, $|\mathrm{Aut}(\mathcal{D})| = 3$
$|\mathrm{Aut}(\mathcal{T})| = 3$, $(0, 9, 6, 7, 6, 0)$

	ade	jnq		for	ghp	bil	ckm
	chj	bmp		giq	efk	ano	dlr
fim		hlo	enp	ajk	bqr	cdg	
gnr		dik	coq	beh	alm	fjp	
abc	ipr		gkl		djo	emq	fhn
ejl	gmo		bdf		cin	hkr	apq
dhq	bkn	afg	jmr	clp			eio
kop	flq	cer	ahi	dmn			bgj

The NKTS(18)s can be distinguished up to isomorphism by their type vectors $(a_1, a_2, \ldots, a_6)$, where each entry a_i gives the number of pairs of parallel classes with Type i. These are displayed with the squares. □

Theorem 3. *There is no doubly resolvable* HATS(19). *There are five almost doubly resolvable* HATS(19)*s, up to isomorphism. Of these five, four appear in a pair of orthogonal resolutions of* $K_{19} - G$ *into APCs, where G consists of three vertex-disjoint triangles.*

Proof. The nonexistence of a doubly resolvable HATS(19) follows from the fact that such a system would also be almost doubly resolvable. Yet in each of the almost doubly resolvable systems given next, some APC intersects the short PPC in two blocks, or the two short PPCs coincide.

The 5 classes of HATS(19)s with an orthogonal resolution for the APCs partition into a single self-orthogonal HATS(19), and 2 pairs of nonisomorphic mutually orthogonal resolutions. Each is shown as a 10×10 square; columns specify the resolution, with the last providing the short PPC. Similarly rows specify a resolution, with the last row providing the short PPC. The 9×9 square obtained by removing the last row and column demonstrates the orthogonality of the APCs. The orders of the automorphism groups for each HATS(19), $\mathcal{T}$, and the underlying STS, $\mathcal{S}$, are given.

$|\mathrm{Aut}(\mathcal{S})| = 2$

#H1: $|\mathrm{Aut}(\mathcal{T})| = 1;\ 2^1 3^2 4^4 5^2$ (columns)

#H1: $|\mathrm{Aut}(\mathcal{T})| = 1;\ 2^1 3^2 4^4 5^2$ (rows)

abc		hks	gmq	inr	dop	fjl			
klo		afg	cjn	bmp		dhr	eis		
	lpr	coq	ahi		fns	ekm		bgj	
gps	fiq	dmn	bor	ajk				cel	
dij				efo	alm	bqs	ckr	hnp	
	cms	ejp		hlq	gik	ano	bdf		
	bkn		dls				apq	imo	cfh,egr
enq		bil	fkp	cdg			hjm	ars	
fmr					beh	cip	gln	dkq	jos
	ade, gho				jqr				

$|\mathrm{Aut}(\mathcal{S})| = 3$

#H3: $|\mathrm{Aut}(\mathcal{T})| = 3;\ 2^3 3^6$ (columns)

#H2: $|\mathrm{Aut}(\mathcal{T})| = 3;\ 2^6 4^3$ (rows)

abc	iko	elr		fnq	ghp	djs			
	afg	bkn	hqr	dip	ejm			cos	
nps	ceq	ahi			fkl		dmo	bgj	
imq	hls		ajk	bor		egn	cfp		
		cdg		alm	bqs	fho	ijn	kpr	
fjr			gis		ano	ckm	beh	dlq	
glo			bdf	eks	cir	apq		hmn	
dhk	bmp	joq	cln				ars	efi	
	dnr	fms	eop	chj		bil	gkq		
									ade, gmr, jlp

$|\mathrm{Aut}(\mathcal{S})| = 3$

#H5: $|\mathrm{Aut}(\mathcal{T})| = 3;\ 3^3 5^3 6^3$ (columns)

#H4: $|\mathrm{Aut}(\mathcal{T})| = 3;\ 4^3 5^6$ (rows)

abc	kms	eip			dlr		fho	gnq	
	ade	cjs	gkr	ioq		fln	bmp		
djo	bil	afg	ens			hmr	ckq		
npr			ahi	cdg	bqs	eko		fjm	
ghs		bor		alm	cef		ijn	dkp	
emq		hkl		fps	ano	bgj		cir	
	chn		bdf	ejr	gim	apq		los	
fik	gop	dmn	jlq				ars	beh	
	fqr		cmo	bkn	hjp	dis	egl		
									ajk, clp, dhq

These HATS(19)s can be distinguished up to isomorphism by their *Pasch type* $a_1^{f_1} a_2^{f_2} \cdots a_t^{f_t}$, which indicates that there are exactly f_i APCs that contain exactly a_i blocks that belong to at least one *Pasch configuration* (that is, a set of blocks of the form $\{abc, ade, bdf, cef\}$) in the underlying STS. In particular, $\sum_{i=1}^{t} f_i = (v-1)/2$ and $0 \leq a_i \leq (v-1)/3$ for each $i = 1, 2, \ldots, t$. $\square$

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Table 5
Number of orthogonal (almost) parallel classes.

N	# (HATS)	# (NKTS)
0	48420	20627
1	594623	4100
2	3591338	491
3	14225701	56
4	41632271	25
5	95972345	9
6	181353940	7
7	289111391	1
8	396814420	10
9	476392987	1
10	506400393	
11	481566312	
12	412931302	
13	321488055	
14	228645394	
15	149278909	1
16	89886341	
17	50093875	
18	25938454	
19	12513898	
20	5643459	
21	2380549	
22	943758	
23	352040	
24	124222	
25	41065	
26	12810	
27	3870	
28	1076	
29	315	
30	69	
31	28	
32	6	
33	2	
35	1	

References

[1] R.D. Baker, R.M. Wilson, Nearly Kirkman triple systems, Utilitas Math. 11 (1977) 289–296.
[2] A.E. Brouwer, Two new nearly Kirkman triple systems, Utilitas Math. 13 (1978) 311–314.
[3] C.J. Colbourn, Triple systems, in: C.J. Colbourn, J.H. Dinitz (Eds.), Handbook of Combinatorial Designs, 2nd ed., Chapman and Hall/CRC, Boca Raton, 2007, pp. 58–71.
[4] C.J. Colbourn, A.D. Forbes, M.J. Grannell, T.S. Griggs, P. Kaski, P.R.J. Östergård, D.A. Pike, O. Pottonen, Properties of the Steiner triple systems of order 19, Electron. J. Combin. 17 (2010) #R98.
[5] C.J. Colbourn, A. Rosa, Triple Systems, Oxford University Press, Oxford, 1999.
[6] I.J. Dejter, F. Franek, A. Rosa, A completion conjecture for Kirkman triple systems, Utilitas Math. 50 (1996) 97–102.
[7] P. Kaski, P.R.J. Östergård, The Steiner triple systems of order 19, Math. Comput. 73 (2004) 2075–2092.
[8] P. Kaski, P.R.J. Östergård, Classification Algorithms for Codes and Designs, Springer, Berlin, 2006.
[9] P. Kaski, P.R.J. Östergård, O. Pottonen, L. Kiviluoto, A catalogue of the Steiner triple systems of order 19, Bull. Inst. Combin. Appl. 57 (2009) 35–41.
[10] P. Kaski, P.R.J. Östergård, S. Topalova, R. Zlatarksi, Steiner triple systems of order 19 and 21 with subsystems of order 7, Discrete Math. 308 (2008) 2732–2741.
[11] P. Kaski, O. Pottonen, libexact User's Guide, Version 1.0, HIIT Technical Reports 2008–1, Helsinki Institute for Information Technology HIIT, 2008.
[12] A. Kotzig, A. Rosa, Nearly Kirkman systems, Congr. Numer. 10 (1974) 607–614.
[13] R. Mathon, A. Rosa, Nearly Kirkman triple systems of order 18, J. Combin. Math. Combin. Comput. 39 (2001) 79–91.
[14] B.D. McKay, Nauty User's Guide (Version 1.5), Technical Report TR-CS-90-02, Computer Science Department, Australian National University, Canberra, 1990.
[15] S. Niskanen, P.R.J. Östergård, Cliquer User's Guide, Version 1.0, Technical Report T48, Communications Laboratory, Helsinki University of Technology, 2003.
[16] R. Rees, D.R. Stinson, On resolvable group-divisible designs with block size 3, Ars Combin. 23 (1987) 107–120.
[17] A. Rosa, Room squares generalized, Ann. Discrete Math. 8 (1980) 43–57.
[18] P. Smith, A doubly divisible nearly Kirkman system, Discrete Math. 18 (1977) 93–96.
[19] S.A. Vanstone, Doubly resolvable designs, Discrete Math. 29 (1980) 77–86.
[20] S.A. Vanstone, D.R. Stinson, P.J. Schellenberg, A. Rosa, R. Rees, C.J. Colbourn, M.J. Carter, J.A. Carter, Hanani triple systems, Israel J. Math. 83 (1993) 305–319.